Chapter 4 Fourier Optics

- Based on harmonic analysis (Fourier transform) and liner system (superposition).
- An arbitrary function

$$f(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\nu_x,\nu_y)\exp[-j2\pi(\nu_x x+\nu_y y)]d\nu_x d\nu_y$$

→ Superposition, or integral of harmonic functions of x and y.

$F(\nu_x,\nu_y)$: Complex amplitude

ν_x,ν_y: Spatial frequency (cycles/unit length)

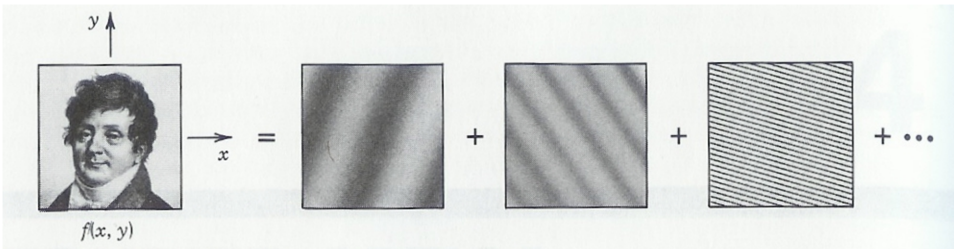

Figure 4.0-2 An arbitrary function $f(x, y)$ may be analyzed as a sum of harmonic functions of different spatial frequencies and complex amplitudes.

- Compare this with plane wave

$$U(x,y,z)=A\exp[-j(k_x x+k_y y+k_z z)]$$

$$U(x,y,0)=A\exp[-j2\pi(\nu_x x+\nu_y y)]$$

$$\nu_x \leftrightarrow k_x/2\pi,\quad \nu_y \leftrightarrow k_y/2\pi$$

- An arbitrary function can be analyzed as a superposition of harmonic functions. → An arbitrary traveling wave $U(x,y,z)$ may be analyzed as a sum of plane waves!

4.1 Propagation of Light in Free Space

A. Correspondence Between the Spatial Harmonic Function and the Plane Wave

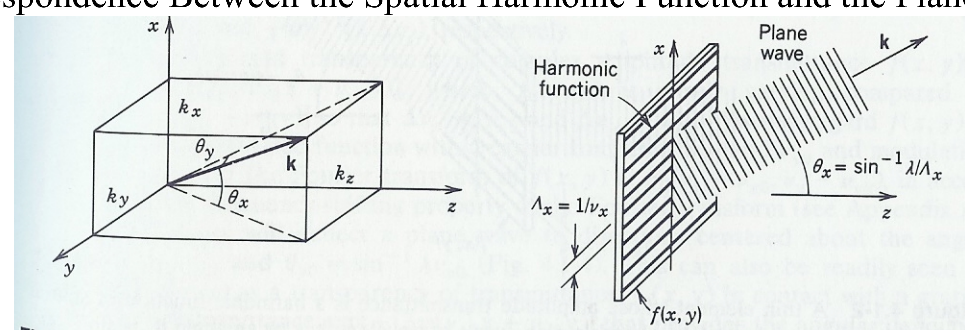

Figure 4.1-1 A harmonic function of spatial frequencies ν_x and ν_y at the plane $z = 0$ is consistent with a plane wave traveling at angles $\theta_x = \sin^{-1}\lambda\nu_x$ and $\theta_y = \sin^{-1}\lambda\nu_y$.

$$\theta_x = \sin^{-1}\left(k_x \big/ k \right) = \sin^{-1}(\lambda \nu_x)$$
$$\theta_y = \sin^{-1}\left(k_y \big/ k \right) = \sin^{-1}(\lambda \nu_y) \qquad (4.1\text{-}1)$$

A physical way of picturing the spatial harmonic function is to project a plane wave on the *x-y* plane.

$$\Lambda_x = 1 \big/ \nu_x, \quad \Lambda_y = 1 \big/ \nu_y$$

$$\rightarrow \qquad \theta_x = \sin^{-1}\left(\lambda \big/ \Lambda_x \right), \quad \theta_y = \sin^{-1}\left(\lambda \big/ \Lambda_y \right)$$

Paraxial approximation:

$$\theta_x = \lambda \big/ \Lambda_x = \lambda \nu_{\mathrm{x}}, \quad \theta_y = \lambda \big/ \Lambda_y = \lambda \nu_{\mathrm{y}} \qquad (4.1\text{-}2)$$

<u>Spatial spectral analysis</u>
(Response of a plane wave after a thin optical element.)
Consider a simple case:

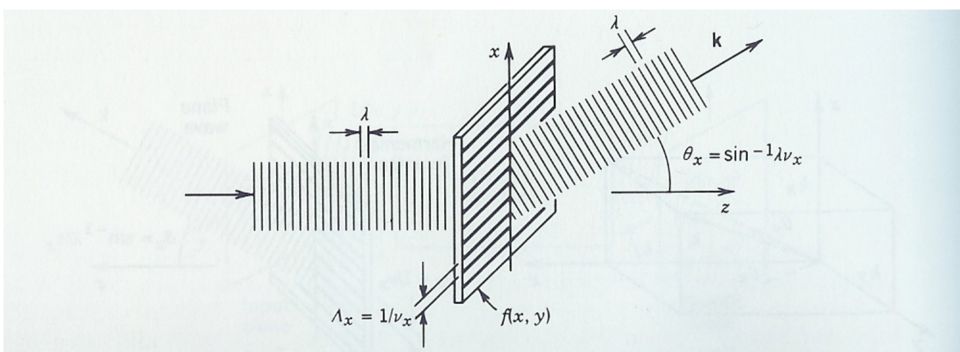

Figure 4.1-2 A thin element whose amplitude transmittance is a harmonic function of spatial frequency ν_x (period $\Lambda_x = 1/\nu_x$) bends a plane wave of wavelength λ by an angle $\theta_x = \sin^{-1} \lambda\nu_x = \sin^{-1}(\lambda/\Lambda_x)$.

$$t(x, y) = \exp[-j2\pi(\nu_x x + \nu_y y)]$$

$\rightarrow$ Harmonic function on *x-y* plane with period $\Lambda_x = 1 \big/ \nu_x$, $\Lambda_y = 1 \big/ \nu_y$.

$$U(x, y, z) = A \exp[-j2\pi(\nu_x x + \nu_y y)] \exp(-jk_z z)$$

$\rightarrow$ Output wave is bent with angles $\theta_x = \sin^{-1}(\lambda \nu_x)$, $\theta_y = \sin^{-1}(\lambda \nu_y)$.
The harmonic function pattern works like a grating.

Now consider a general case:

$$t(x, y) = \iint F(\nu_x, \nu_y) \exp[-j2\pi(\nu_x x + \nu_y y)] d\nu_x d\nu_y \qquad (4.1\text{-}4)$$

$$U(x,y,z) = \iint F(\nu_x,\nu_y)\exp[-j2\pi(\nu_x x+\nu_y y)]\exp(-jk_z z)d\nu_x d\nu_y$$

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} = 2\pi\sqrt{\frac{1}{\lambda^2} - \frac{1}{\lambda_x^2} - \frac{1}{\lambda_y^2}}$$

→ An incident plane wave is decomposed into many plane waves, each traveling at angles $\theta_x = \sin^{-1}(\lambda\nu_x)$, $\theta_y = \sin^{-1}(\lambda\nu_y)$, with a complex envelope $F(\nu_x,\nu_y)$, the Fourier transform of $f(x,y)$.

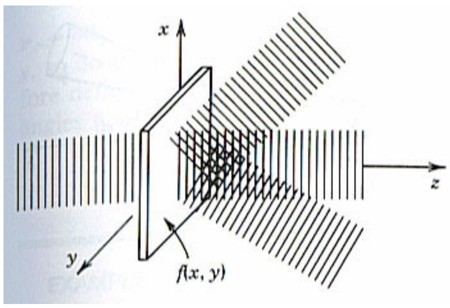

Figure 4.1-3 A thin optical element of amplitude transmittance $f(x,y)$ decomposes an incident plane wave into many plane waves. The plane wave traveling at the angles $\theta_x = \sin^{-1}\lambda\nu_x$ and $\theta_y = \sin^{-1}\lambda\nu_y$ has a complex envelope $F(\nu_x,\nu_y)$, the Fourier transform of $f(x,y)$.

Example 4.1-2, Imaging

$$t(x,y) = \exp\left(j\frac{\pi x^2}{\lambda f}\right) = \exp\left(-j2\pi\varphi(x,y)\right)$$

$$\varphi(x,y) = -\frac{x^2}{2\lambda f}$$

Compare to earlier: $\varphi(x,y) \leftrightarrow \nu_x x + \nu_y y$

Now ν_x varies with $x \Rightarrow \nu_x = \dfrac{\partial\varphi(x,y)}{\partial x} = -\dfrac{x}{\lambda f}$

$$\Rightarrow \theta_x = \sin^{-1}\left(-\frac{x}{f}\right)$$

→ A cylindrical lens with focal length *f*

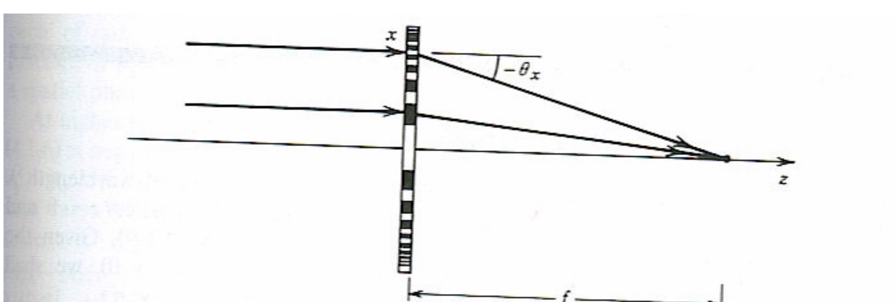

Figure 4.1-7 A transparency with transmittance $f(x,y) = \exp(j\pi x^2/\lambda f)$ bends the wave at position x by an angle $\theta_x \approx -x/f$ so that it acts as a cylindrical lens with focal length f.

B. Transfer Function of Free Space

Since an arbitrary function can be analyzed as sum of harmonic functions, we consider a harmonic input function.

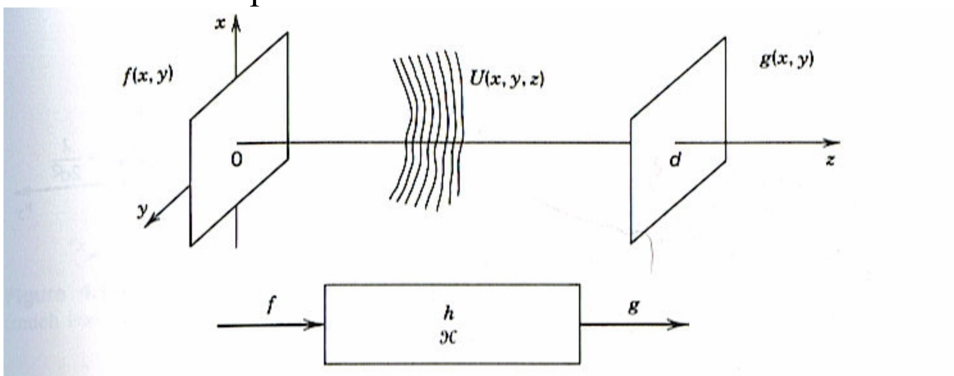

Figure 4.1-9 Propagation of light between two planes is regarded as a linear system whose input and output are the complex amplitudes of the wave in the two planes.

$$f(x,y) = U(x,y,0) = A\exp[-j2\pi(\nu_x x + \nu_y y)]$$

Output $\quad g(x,y) = U(x,y,d) = A\exp[-j(k_x x + k_y y + k_z d)]$

$$\mathrm{H}(\nu_x,\nu_y) = \frac{g(x,y)}{f(x,y)} = \exp(-jk_z d)$$

$$= \exp\left[-j2\pi\left(\frac{1}{\lambda^2} - \nu_x^2 - \nu_y^2\right)^{1/2} d\right] \qquad (4.1\text{-}6)$$

<u>Fresnel approximation</u>

$$\nu_x^2 + \nu_y^2 << 1/\lambda^2$$

→ The plane-wave components of the propagating light make small angles $\theta_x \sim \lambda\nu_x, \theta_y \sim \lambda\nu_y$.

→ Paraxial waves:

$$H(\nu_x,\nu_y) = \exp(-jkd)\exp\left[j\pi\lambda d\left(\nu_x^2 + \nu_y^2\right)\right] \qquad (4.1\text{-}8)$$

Validity of Fresnel approximation has the same expression as in Sec. 2.2.

<u>Input-output relation</u>

Given the input function $f(x,y)$, how to obtain the output $g(x,y)$:

(1) Determine the complex envelopes of the plane-wave components in the input plane by Fourier transform.

$$F(\nu_x,\nu_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[j2\pi(\nu_x x + \nu_y y)]dxdy$$

(2) Complex envelopes of the plane-wave components in the output plane =

$$H(\nu_x,\nu_y)F(\nu_x,\nu_y)$$

(3) $$g(x,y)=\iint H(\nu_x,\nu_y)F(\nu_x,\nu_y)\exp[-j2\pi(\nu_x x+\nu_y y)]d\nu_x d\nu_y$$

Under Fresnel approximation,

$$g(x,y)=H_0\iint F(\nu_x,\nu_y)\exp[j\pi\lambda d(\nu_x^2+\nu_y^2)]\exp[-j2\pi(\nu_x x+\nu_y y)]d\nu_x d\nu_y$$

$$H_0\equiv\exp(-jkd)$$

<u>Free-space propagation as a convolution</u>

Each point generates a spherical wave. Under Fresnel approximation (observation point close to the propagation axis), spherical wave → parabolic wave.

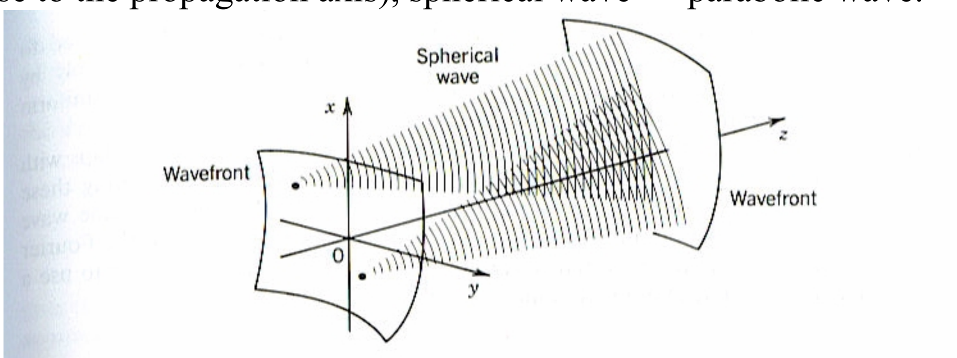

Figure 4.1-12 The Huygens–Fresnel principle. Each point on a wavefront generates a spherical wave.

$$h(x,y)\approx h_0\exp\left[-jk\frac{x^2+y^2}{2d}\right] \qquad (4.1\text{-}13)$$

$$h_0=\frac{j}{\lambda d}\exp(-jkd)$$

$$g(x,y)=h_0\iint f(x',y')\exp\left[-j\pi\frac{(x-x')^2+(y-y')^2}{\lambda d}\right]dx'dy' \qquad (4.1\text{-}14)$$

4.2 Optical Fourier Transform

A plane wave transmitting through an optical element can be used to decompose the harmonic functions (Fourier components $F(\nu_x,\nu_y)$) that compose the pattern ($f(x,y)$) on the optical element.

A. Fourier Transform in the Far Field (Fraunhofer Approximation)

If $f(x,y)$ is confined to a small area of radius *b*, distance d to the observation plane is sufficiently large, so that Fresnel number for $f(x,y)$, $N_F'=b^2/\lambda d<<1$.

$$g(x,y) = h_0 \exp\left(-j\pi \frac{x^2+y^2}{\lambda d}\right) F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right) \qquad (4.2\text{-}4)$$

Furthermore, if we limit our interest to points at the output plane within a circle of radius a centered about the z axis, so that $N_F = a^2/\lambda d \ll 1$ for $g(x,y)$.

$$g(x,y) = h_0 F\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right) \qquad (4.2\text{-}1)$$

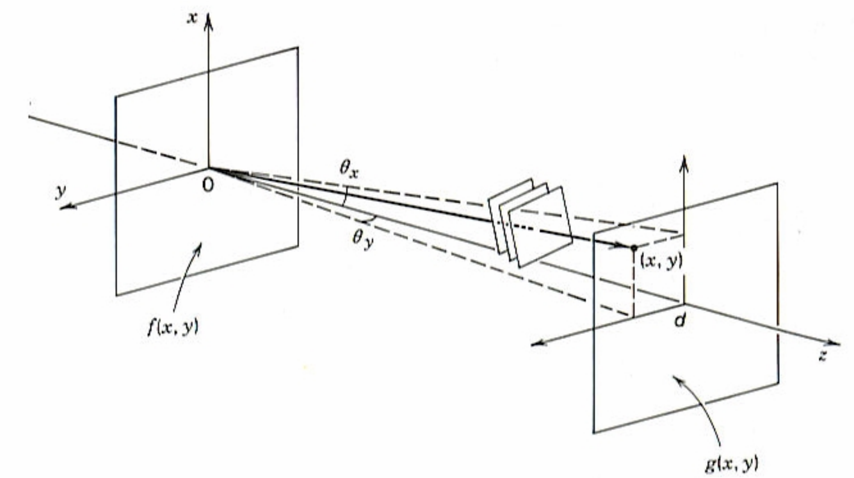

Figure 4.2-1 When the distance d is sufficiently long, the complex amplitude at point (x, y) in the $z = d$ plane is proportional to the complex amplitude of the plane-wave component with angles $\theta_x \approx x/d \approx \lambda\nu_x$ and $\theta_y \approx y/d \approx \lambda\nu_y$, i.e., to the Fourier transform $F(\nu_x, \nu_y)$ of $f(x, y)$, with $\nu_x = x/\lambda d$ and $\nu_y = y/\lambda d$.

→ The only plane wave that contributes to the complex amplitude at (x, y) at output plane is the wave making angles $\theta_x = x/d, \theta_y = y/d$ with the optical axis. This is also the wave with wave-vector components $k_x = \left(x/d\right)k, k_y = \left(y/d\right)k$ and amplitude $F(\nu_x, \nu_y)$ with $\nu_x = x/\lambda d, \nu_y = y/\lambda d$.

- Fraunhofer approximation is valid when both N_F and N_F' are small.

B. Fourier Transform Using a Lens

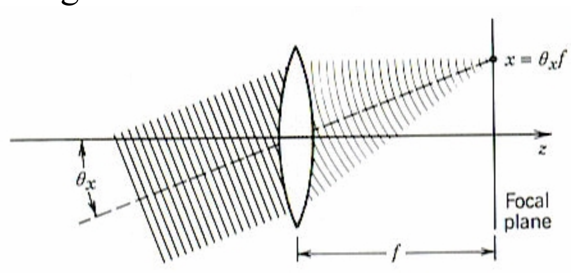

Figure 4.2-2 Focusing of a plane wave into a point. A direction (θ_x, θ_y) is mapped into a point $(x, y) = (\theta_x f, \theta_y f)$.

Amplitude of the plane wave with direction $(\theta_x, \theta_y) = (\lambda \nu_x, \lambda \nu_y)$ is proportional to the Fourier transform $F(\nu_x, \nu_y)$ and is located at the point $(x, y) = (\theta_x f, \theta_y f) = (\lambda f \nu_x, \lambda f \nu_y)$.

$$\rightarrow \quad g(x,y) \propto F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \qquad (4.2\text{-}5)$$

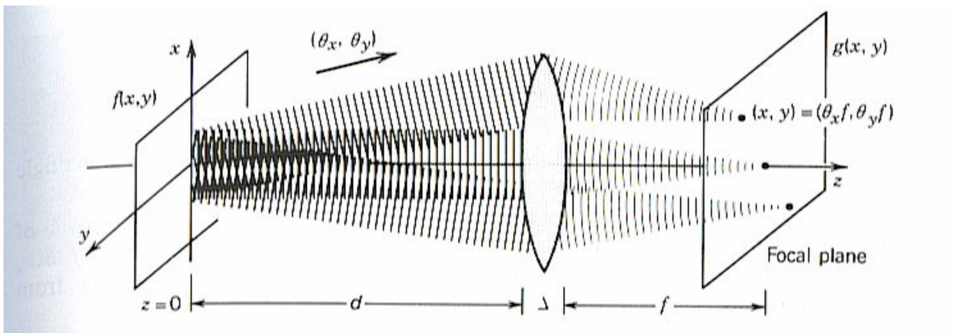

Figure 4.2-3 Focusing of the plane waves associated with the harmonic Fourier components of the input function $f(x, y)$ into points in the focal plane. The amplitude of the plane wave with direction $(\theta_x, \theta_y) = (\lambda \nu_x, \lambda \nu_y)$ is proportional to the Fourier transform $F(\nu_x, \nu_y)$ and is focused at the point $(x, y) = (\theta_x f, \theta_y f) = (\lambda f \nu_x, \lambda f \nu_y)$.

$$g(x,y) = \frac{j}{\lambda f} \exp[-jk(d+f)] \exp\left[j\pi \frac{(x^2+y^2)(d-f)}{\lambda f^2} \right] F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \qquad (4.2\text{-}8)$$

$$I(x,y) = \frac{1}{|\lambda f|^2} \left| F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right) \right|^2 \qquad (4.2\text{-}9)$$

If $d = f$, $g(x,y) = \frac{j}{\lambda f} \exp[-j2kf] F\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right)$ (4.2-10)

Fourier transform using a lens is valid in Fresnel approximation (only radius at the output is limited). Without the lens, we need Fraunhofer approximation (radii at both output and input are limited).

4.3 Diffraction of Light

Light not simply blocked by an opaque object, as in Ray Optics. It depends on the wavelength, the dimension of the object, and the distance between the object and the observation plane.

A. Fraunhofer Diffraction

Aperture function $p(x, y)$, with Fourier components $P(\nu_x, \nu_y) = P\left(\frac{x}{\lambda d}, \frac{y}{\lambda d}\right)$.

Assume the incident wave is a plane wave of intensity I_i in z-direction.
Using Eq. (4.2-1), Fraunhofer approximation, we obtain:

$$I(x,y) = \frac{I_i}{(\lambda d)^2}\left|P\left(\frac{x}{\lambda d},\frac{y}{\lambda d}\right)\right|^2 \tag{4.3-4}$$

→ Proportional to the squared magnitude of the Fourier transform of the aperture function $p(x,y)$ evaluated at the spatial frequency $\nu_x = \frac{x}{\lambda d}, \nu_y = \frac{y}{\lambda d}$.

<u>Example: Fraunhofer diffraction from a circular aperture</u>

$$I(\rho) = \left(\frac{\pi D^2}{4\lambda d}\right)^2 I_i \left[\frac{2J_1(\pi D\rho/\lambda d)}{\pi D\rho/\lambda d}\right]^2 \tag{4.3-7}$$

→ Airy pattern. Center disk (Airy disk) has radius $\rho_s = 1.22\lambda d / D$, subtending an angle $\theta = 1.22\lambda / D$.

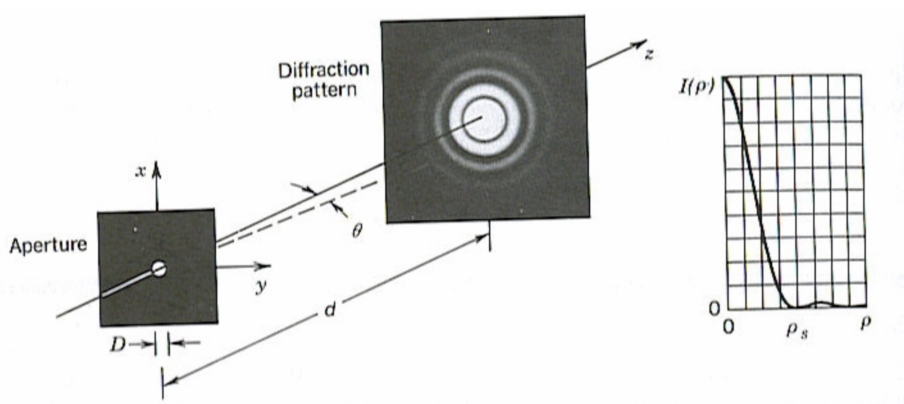

Figure 4.3-4 The Fraunhofer diffraction pattern from a circular aperture produces the Airy pattern with the radius of the central disk subtending an angle $\theta = 1.22\lambda/D$.

B. Fresnel Diffraction

At small distance ($d \to 0$), the diffraction pattern is the shadow of the aperture. At medium distance (Fresnel diffraction), the diffraction pattern is the convolution of the aperture. Using Eq. (4.1-14), free-space propagation as a convolution, we obtain:

$$I(x,y) = \frac{I_i}{(\lambda d)^2}\left|\iint p(x',y')\exp\left[-j\pi\frac{(x-x')^2+(y-y')^2}{\lambda d}\right]dx'dy'\right|^2 \tag{4.3-11}$$

At large *d*, the diffraction pattern becomes Fraunhofer diffraction pattern. The far field has an angular divergence proportional to λ / D, where D is the diameter of the aperture.

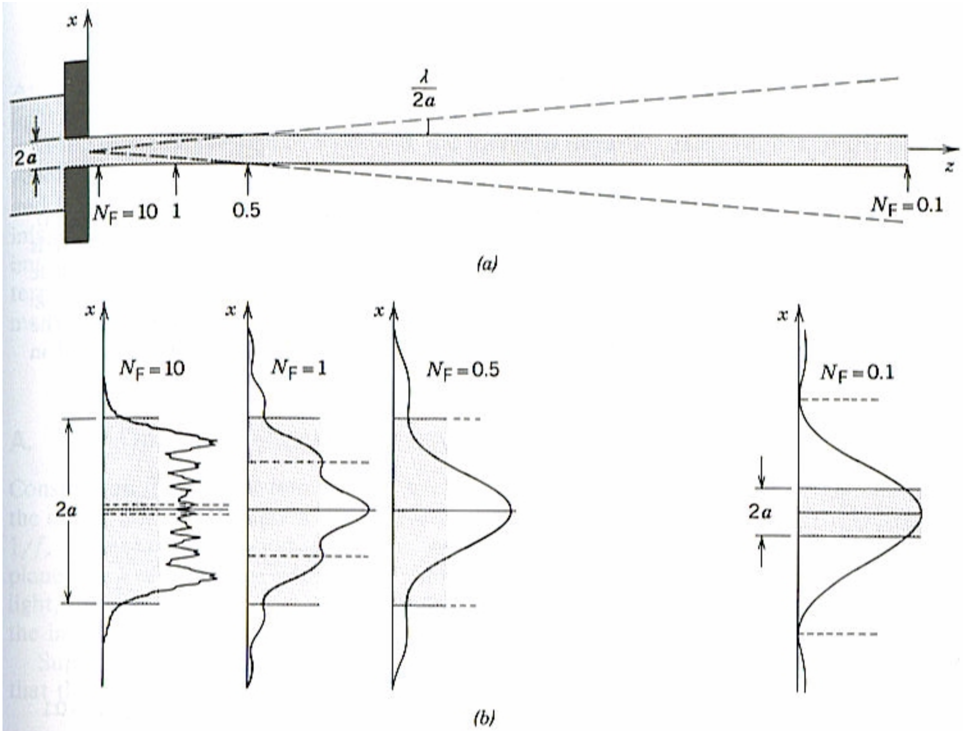

Figure 4.3-7 Fresnel diffraction from a slit of width $D = 2a$. (a) Shaded area is the geometrical shadow of the aperture. The dashed line is the width of the Fraunhofer diffracted beam. (b) Diffraction pattern at four axial positions marked by the arrows in (a) and corresponding to the Fresnel numbers $N_F = 10, 1, 0.5$, and 0.1. The shaded area represents the geometrical shadow of the slit. The dashed lines at $|x| = (\lambda/D)d$ represent the width of the Fraunhofer pattern in the far field. Where the dashed lines coincide with the edges of the geometrical shadow, the Fresnel number $N_F = a^2/\lambda d = 0.5$.

4.4 Image Formation

<u>Spatial filtering</u>

Two-lens imaging system (4-*f* system). Unity maginification.

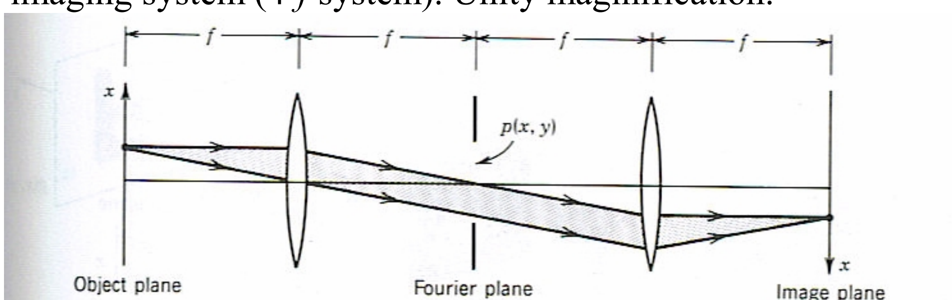

Figure 4.4-3 The 4–f imaging system. If an inverted coordinate system is used in the image plane, the magnification is unity.

4-*f* imaging system for Fourier transform. The Fourier components of $f(x, y)$ are separated by the lens. Each point in the Fourier plane corresponds to a single spatial frequency (Recall Fig. 4.2-2). The second lens reconstructs the image.

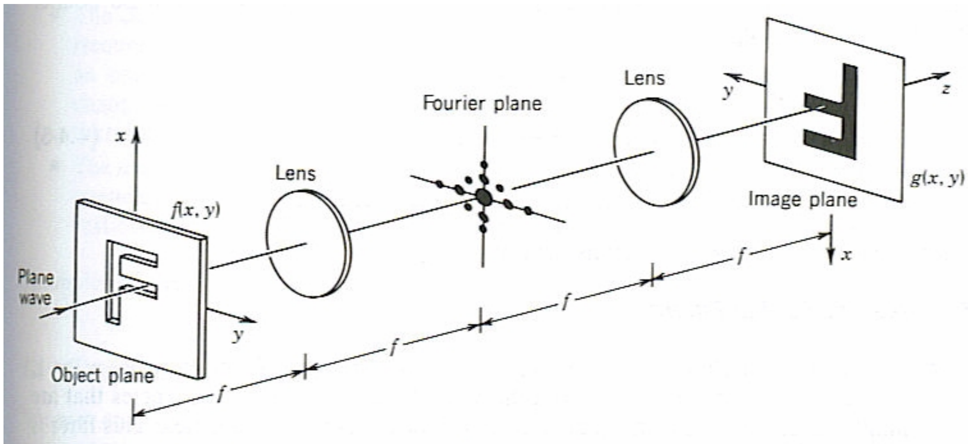

Figure 4.4-4 The 4–f system performs a Fourier transform followed by an inverse Fourier transform, so that the image is a perfect replica of the object.

Spatial filtering: Add a mask at the Fourier plane to block unwanted Fourier components of $f(x, y)$.

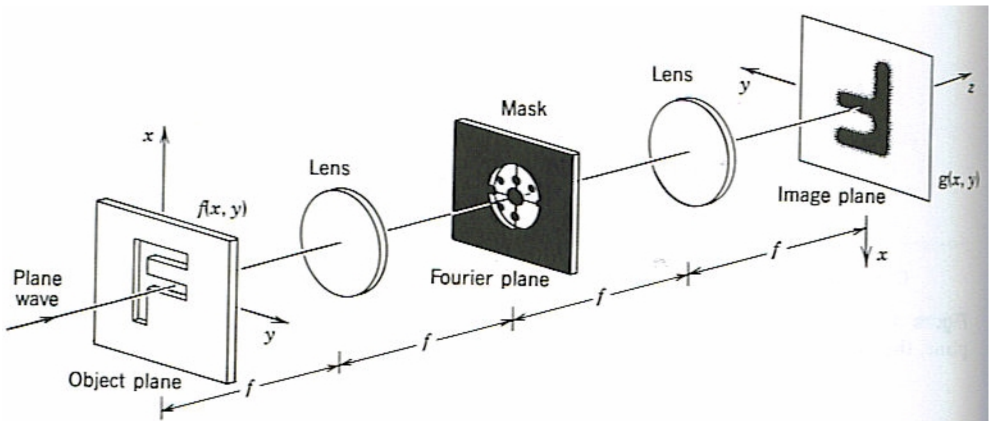

Figure 4.4-5 Spatial filtering. The transparencies in the object and Fourier planes have complex amplitude transmittances $f(x, y)$ and $p(x, y)$. A plane wave traveling in the z direction is modulated by the object transparency, Fourier transformed by the first lens, multiplied by the transmittance of the mask in the Fourier plane and inverse Fourier transformed by the second lens. As a result, the complex amplitude in the image plane $g(x, y)$ is a filtered version of $f(x, y)$. The system has a transfer function $\mathcal{H}(\nu_x, \nu_y) = p(\lambda f \nu_x, \lambda f \nu_y)$.

Transfer function of the mask for the Fourier components:

$$H(\nu_x, \nu_y) = p(\lambda f \nu_x, \lambda f \nu_y) \qquad (4.4\text{-}4)$$

Output: $G(\nu_x, \nu_y) = H(\nu_x, \nu_y) F(\nu_x, \nu_y)$

Example:

(a) Low-pass filter

$$H(\nu_x, \nu_y) = 1 \text{ for } \nu_x^2 + \nu_y^2 < \nu_s^2, \qquad \nu_s: \text{cutoff frequency}$$

$$H(\nu_x, \nu_y) = 0 \text{ othewise}$$

A low-pass filter for spatial frequency is a circular aperture of diameter $D = 2\nu_s \lambda f$.

(b) High-pass filter

Complement of low-pass filter.

Output is high at regions of large rate of change, small at regions of smooth or slow variation of the object.

Application: Edge enhancement in image-processing.

(c) Vertical-pass filter

Blocks horizontal frequency and transmits vertical frequency.

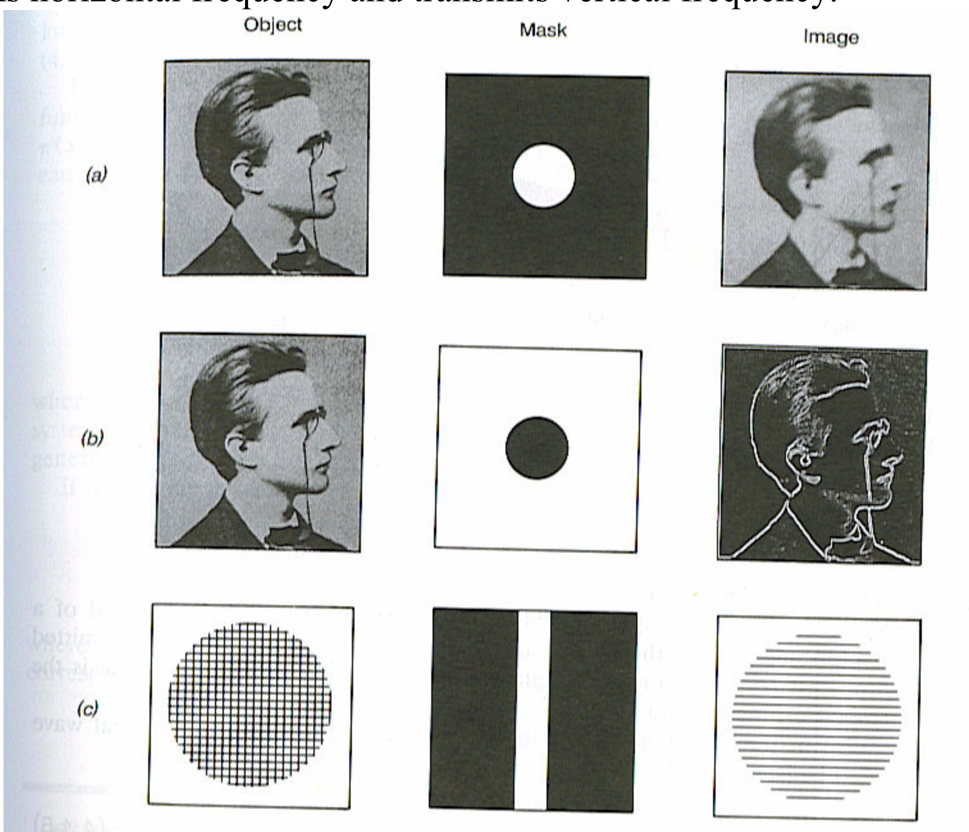

Figure 4.4-6 Examples of object, mask, and filtered image for three spatial filters: (*a*) low-pass filter; (*b*) high-pass filter; (*c*) vertical-pass filter. Black means the transmittance is zero and white means the transmittance is unity.

4.5 Holography

Recording and reconstruction of optical waves.

Consider an arbitrary monochromatic optical wave. At $z = 0$ plane, $U = U_0(x, y)$. If a thin optical element (transparency) has complex amplitude transmittance $t(x, y) = U_0(x, y)$. Illuminate the transparency with a uniform plane wave in z-direction, the optical wave $U(x, y)$ can be reconstructed.

Transparency → Hologram

How to get $t(x,y)$ from $U_0(x,y)$? Phase information is very important. Need some kind of coding to transform phase into intensity.

Holographic code and off-axis holography

Mixing the original wave (object wave) U_0 with a known reference wave U_r, and recording their interference pattern in $z=0$ plane.

$$t(x,y) \propto I_r + I_0 + U_r^* U_0 + U_r U_0^* \tag{4.5-1}$$

Decoding: Illuminate the hologram with U_r,

$$U = tU_r \propto U_r I_r + U_r I_0 + I_r U_0 + U_r^2 U_0^* \tag{4.5-2}$$

First and second terms → Reference wave
Third term → Original wave
Fourth term → Conjugate of the original wave

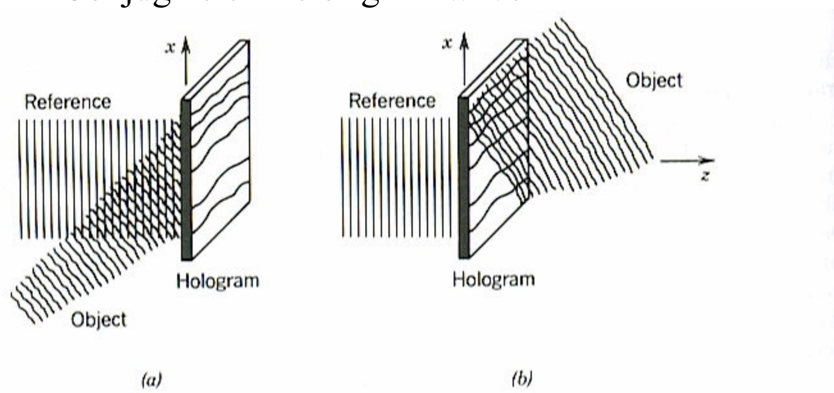

Figure 4.5-1 (*a*) A hologram is a transparency on which the interference pattern between the original wave (object wave) and a reference wave is recorded. (*b*) The original wave is reconstructed by illuminating the hologram with the reference wave.

Example 4.5-1: Hologram of an oblique plane wave

$$U_0(x,y) = \sqrt{I_0}\exp(-jkx\sin\theta)$$

$$U(x,u) \propto I_r + I_0 + \sqrt{I_r I_0}\exp(-jkx\sin\theta) + \sqrt{I_r I_0}\exp(jkx\sin\theta)$$

→ Diffraction grating

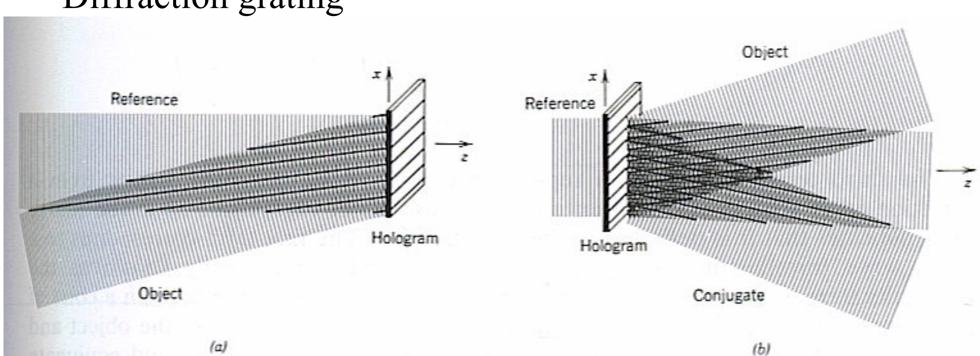

Figure 4.5-2 The hologram of an oblique plane wave is a sinusoidal diffraction grating: (*a*) recording; (*b*) reconstruction.

How to make sure the object wave can be well separated?
Consider an arbitrary object wave whose propagation direction centers about θ. At $z=0$ plane, $U_0(x,y)=f(x,y)\exp(-jkx\sin\theta)$.

Assume $f(x,y)$ varies slowly so that its maximum spatial frequency ν_s, corresponding to $\theta_s=\sin^{-1}(\lambda\nu_s)<<\theta$.

$$U(x,y)\propto I_r+|f(x,y)|^2+\sqrt{I_r}f(x,y)\exp(-jkx\sin\theta)+\sqrt{I_r}f^*(x,y)\exp(+jkx\sin\theta)$$

$|f(x,y)|^2$: Ambiguity term → Non-uniform plane wave in directions with a cone of $2\theta_s$ around the z-direction.

If (1) $\theta>3\theta_x$, or (2) $I_r>>|f(x,y)|^2$, the original wave can be resolved unambiguously.

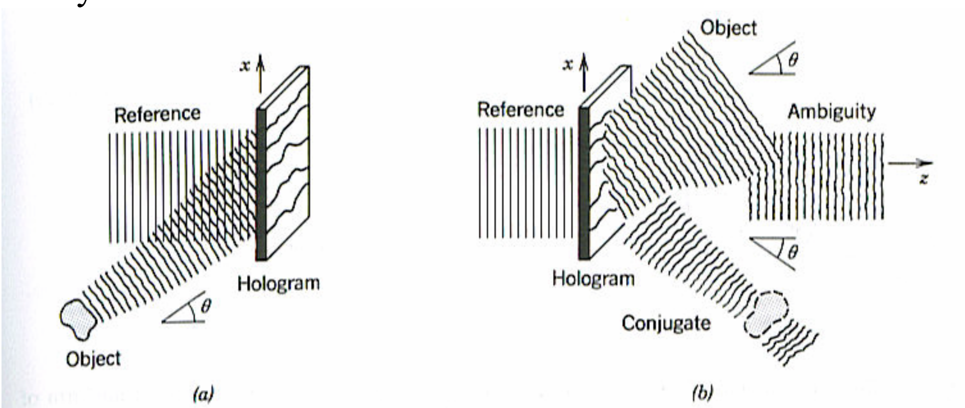

Figure 4.5-4 Hologram of an off-axis object wave: (*a*) recording; (*b*) reconstruction. The object wave is separated from both the reference and conjugate waves.

<u>Fourier-transform holography</u>

Fourier transform $F(\nu_x,\nu_y)$ of a function $f(x,y)$ can be obtained by a lens (see Sec. 4.2). $F(\nu_x,\nu_y)=F(x/\lambda f, y/\lambda f)=U_0(x,y)$.

Illuminate the hologram with U_r reconstruct F. The original function $f(x,y)$ is reconstructed at the focal plane using a lens.

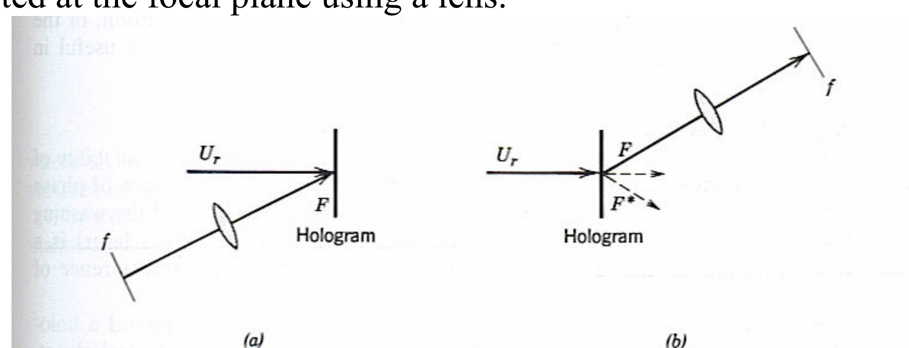

Figure 4.5-5 Hologram of a wave whose complex amplitude represents the Fourier transform of a function $f(x, y)$: (*a*) recording; (*b*) reconstruction.

The holographic apparatus

One essential requirement for holography: A monochromatic light source with minimum phase fluctuations.

→ A coherent light source, usually a laser.

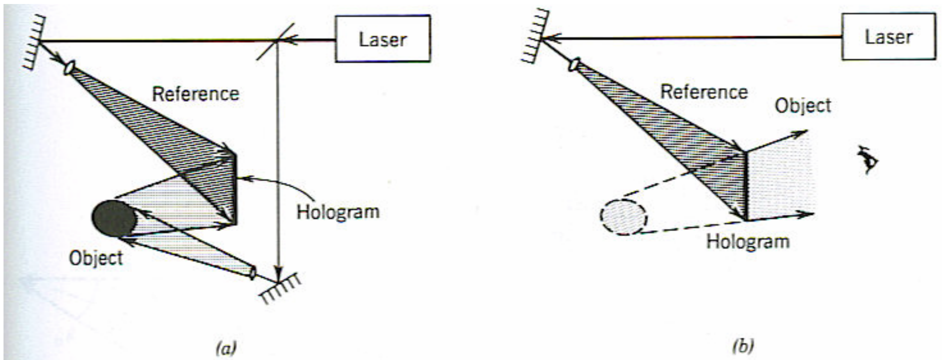

Figure 4.5-7 Holographic recording (a) and reconstruction (b).