Chapter 2 Wave Optics

- Improves on Ray Optics by including phenomena such as interference and diffraction.
- Limitations: (1) Cannot provide a complete picture of reflection and refraction at the boundaries between dielectric materials. (2) Cannot explain optical phenomena that require vector formalism, such as polarization.

2.1 Postulates of Wave Optics

The wave equation

Light speed in a medium: $c = \dfrac{c_0}{n}$ (2.1-1)

Wave function $u(\vec{r},t)$ [position $\vec{r} = (x,y,z)$, time t] satisfies

Wave equation $\nabla^2 u - \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = 0$ (2.1-2)

Wave equation is linear. Principle of *superposition* applies: $u(\vec{r},t) = u_1(\vec{r},t) + u_2(\vec{r},t)$

Wave equation approximately applicable to media with position-dependent refractive indices, provided that the variation is slow within distances of a wavelength → Locally homogeneous, $n = n(\vec{r})$, $c = c(\vec{r})$.

Intensity, power, and energy

Optical intensity: optical power per unit area (watts/cm^2)

$$I(\vec{r},t) = 2\left\langle u^2(\vec{r},t)\right\rangle \qquad (2.1\text{-}3)$$

$\langle\ \rangle$: average over a period >> 1/frequency.

Optical power (watts) flowing into an area A normal to the propagation direction:

$$P(t) = \int_A I(\vec{r},t)\,dA \qquad (2.1\text{-}4)$$

Optical energy (joules) collected in a given time interval T is $\int_T P(t)\,dt$.

2.2 Monochromatic Waves

$$u(\vec{r},t) = a(\vec{r})\cos\left[2\pi f t + \varphi(\vec{r})\right] \qquad (2.2\text{-}1)$$

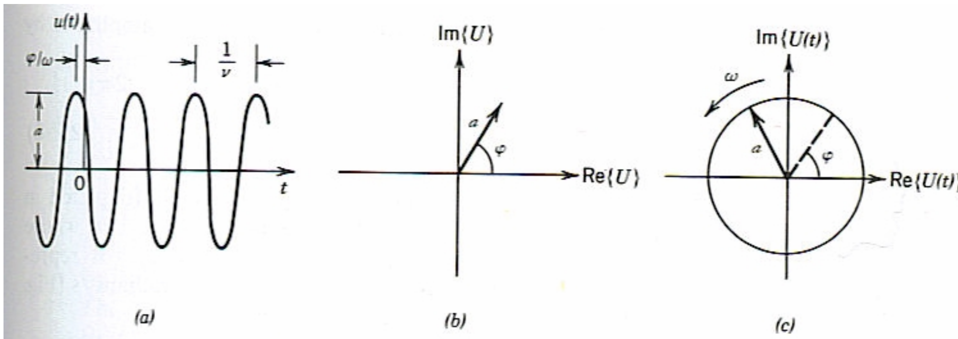

Figure 2.2-1 Representations of a monochromatic wave at a fixed position **r**: (*a*) the wavefunction $u(t)$ is a harmonic function of time; (*b*) the complex amplitude $U = a\exp(j\varphi)$ is a fixed phasor; (*c*) the complex wavefunction $U(t) = U\exp(j2\pi\nu t)$ is a phasor rotating with angular velocity $\omega = 2\pi\nu$ radians/s.

A. Complex Representation and the Helmholtz Equation

Complex wavefunction

$$U(\vec{r},t) = a(\vec{r})\exp[j\varphi(\vec{r})]\exp(j2\pi ft) = U(\vec{r})e^{j\omega t} \qquad (2.2\text{-}2,5)$$

$U(\vec{r})$: complex amplitude

$$\nabla^2 U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = 0 \qquad (2.2\text{-}4)$$

The Helmholtz equation and wavenumber

Substituting (2.2-5) into (2.2-4) obtains:

$$(\nabla^2 + k^2)U(\vec{r}) = 0 \qquad (2.2\text{-}7)$$

$$k = \omega/c \quad \text{: wavenumber} \qquad (2.2\text{-}8)$$

Optical intensity

$$I(\vec{r}) = \left|U(\vec{r})\right|^2 \qquad (2.2\text{-}10)$$

not a function of time.

Wavefronts

The wavefronts are the surfaces of equal phase, $\varphi(\vec{r}) =$ constant.

B. Elementary Waves

The plane wave

Complex amplitude:

$$U(\vec{r}) = A\exp(-j\vec{k}\cdot\vec{r}) = A\exp[-j(k_x x + k_y y + k_z z) \qquad (2.2\text{-}11)$$

$\vec{k}$: wavevector → direction of propagation

$\left|\vec{k}\right| = k =$ wavenumber

Wavelength $\lambda = \frac{2\pi}{k} = \frac{c}{f}$ (2.2-12)

c is called the *phase velocity* of the wave.

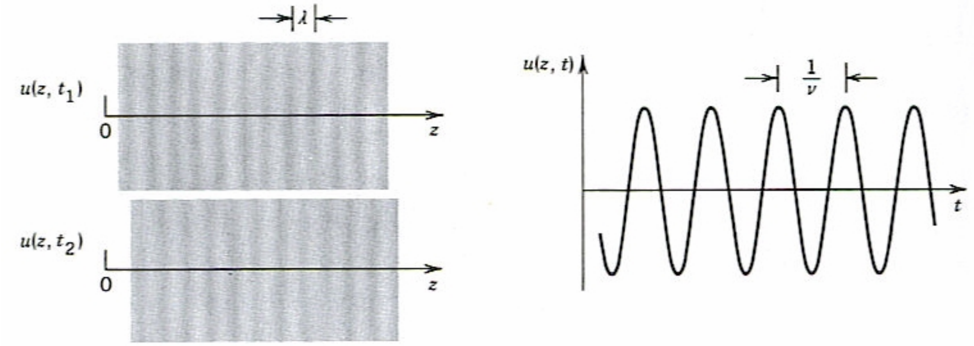

Figure 2.2-2 A plane wave traveling in the z direction is a periodic function of z with spatial period λ and a periodic function of t with temporal period $1/\nu$.

In a medium of refractive index n, f is the same,

$$c = \frac{c_0}{n}, \quad \lambda = \frac{\lambda_0}{n}, \quad k = nk_0 \tag{2.2-14}$$

The spherical wave

$$U(\vec{r}) = \frac{A}{r} e^{-jkr} \tag{2.2-15}$$

$$I(\vec{r}) = \frac{|A|^2}{r^2}$$

The paraboloidal wave

Fresnel approximation of the spherical wave.

Examine a spherical wave originating at $\vec{r} = 0$ at points $\vec{r} = (x, y, z)$ sufficiently close to the z axis but far from the origin, so that $\sqrt{(x^2 + y^2)} << z$. Use paraxial approximation, Taylor series expansion, and Fresnel approximation:

$$U(\vec{r}) = \frac{A}{z} \exp(-jkz) \exp\left(-jk\frac{x^2 + y^2}{2z}\right) \tag{2.2-16}$$

First phase term: planar wave. Second phase term: paraboloidal wave.

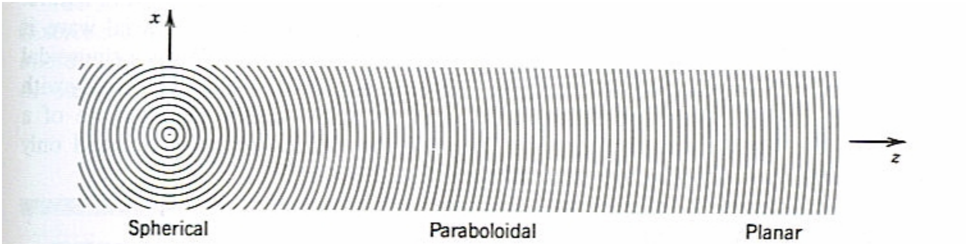

Figure 2.2-4 A spherical wave may be approximated at points near the z axis and sufficiently far from the origin by a paraboloidal wave. For very far points, the spherical wave approaches the plane wave.

- Validity of Fresnel approximation
  Fresnel approximation valid for points (x, y) lying within a circle of radius a centered about the z axis at position z, if a satisfies $a^4 << 4z^3\lambda$, or

$$\frac{N_F \theta_m^2}{4} << 1 \qquad (2.2\text{-}17)$$

where $\theta_m = a/z$ is the maximum angle, and

$$N_F = \frac{a^2}{\lambda z} \quad \text{Fresnel number} \qquad (2.2\text{-}18)$$

2.4 Simple Optical Components

A. Reflection and Refraction

Laws of reflection and refraction can be verified by wave optics. Please read the textbook.

B. Transmission through Optical Components

(Ignore reflection and absorption. Main emphasis on phase shift and associated wavefront bending.)

<u>Transmission through a transparent plate</u>

- Normal incidence

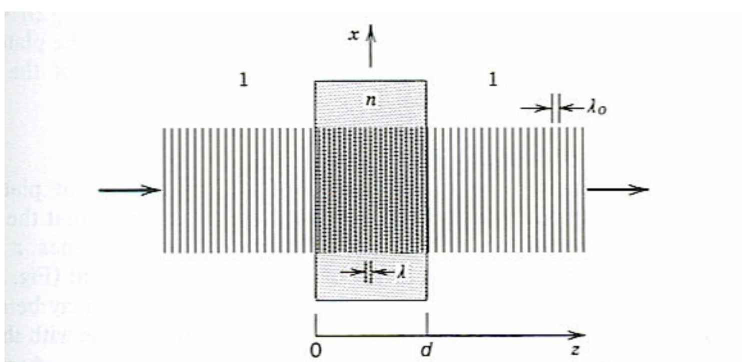

Figure 2.4-3 Transmission of a plane wave through a transparent plate.

Complex amplitude transmittance

$$t(x,y) = \exp(-jnk_0 d) \quad (2.4\text{-}3)$$

→ The plate introduces a phase shift $nk_0 d = 2\pi \dfrac{d}{\lambda}$

- Oblique incidence

$$t(x,y) = \exp\left[-jnk_0(d\cos\theta_1 + x\sin\theta_1)\right]$$

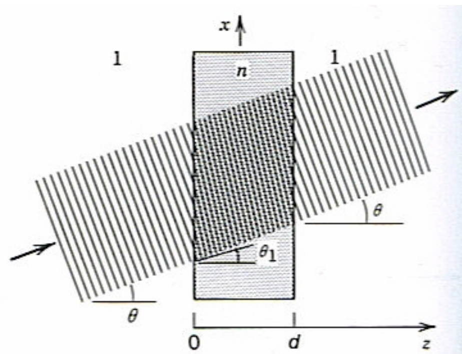

Figure 2.4-4 Transmission of an oblique plane wave through a thin transparent plate.

Thin transparent plate of varying thickness

Transmittance = (Transmittance in air) × (Transmittance in plate)

$$\begin{aligned} t(x,y) &= \exp\left[-jk_0(d_0 - d(x,y))\right]\exp\left[-jnk_0 d(x,y)\right] \\ &= \exp(-jk_0 d_0)\exp\left[-j(n-1)k_0 d(x,y)\right] \end{aligned} \quad (2.4\text{-}4)$$

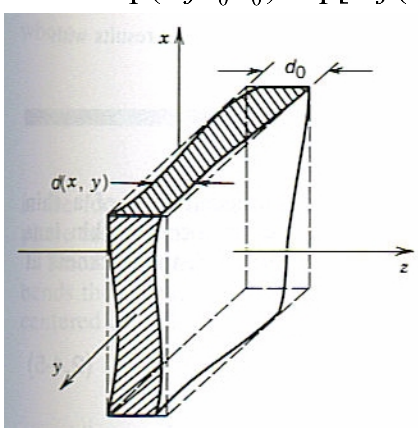

Figure 2.4-5 A transparent plate of varying thickness.

Thin lens

Utilizing Eq. (2.4-4) results in

$$t(x,y) = h_0 \exp\left[jk_0 \frac{x^2+y^2}{2f}\right] \quad (2.4\text{-}6)$$

where $h_0 = \exp(-jnk_0 d_0)$: constant phase factor

$f = \dfrac{R}{n-1}$: focus length of the lens

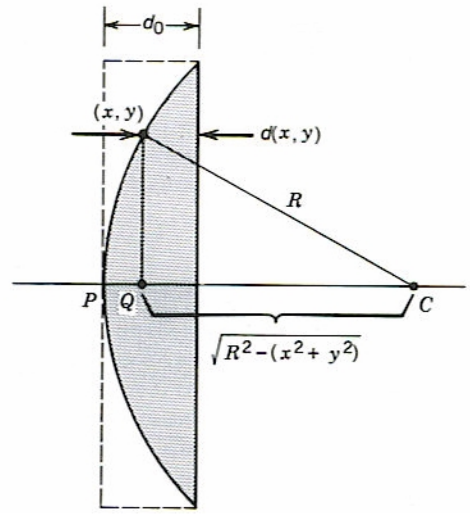

Figure 2.4-7 A planoconvex lens.

<u>Diffraction grating</u>

An optical component periodically modulates the phase or the amplitude of the incident wave. It can be made of a transparent plate with periodically varying thickness or periodically graded refractive index.

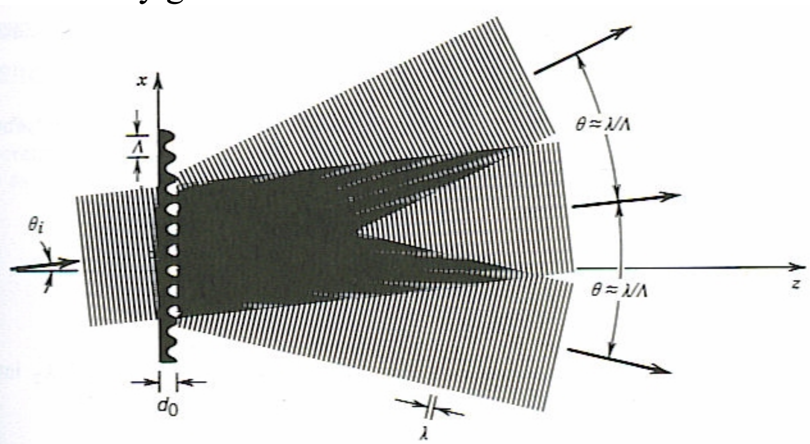

Figure 2.4-11 A thin transparent plate with periodically varying thickness serves as a diffraction grating. It splits an incident plane wave into multiple plane waves traveling in different directions.

The diffraction grating shown above converts an incident plane wave of wavelength $\lambda << \Lambda$, traveling at a small angle θ_i with respect to the *z* axis, into several plane waves at small angles

$$\theta_q = \theta_i + q\frac{\lambda}{\Lambda} \qquad (2.4\text{-}9)$$

with the *z* axis.

q = 0, ±1, ±2, … : diffraction order

In general, without paraxial approximation

$$\sin\theta_q = \sin\theta_i + q\frac{\lambda}{\Lambda} \qquad (2.4\text{-}10)$$

C. Graded-index Optical Components

Instead of varying thickness, varying refractive index.

Varying thickness: $t(x,y) = \exp[-jn_0 k_0 d(x,y)]$

Varying refractive index: $t(x,y) = \exp[-jn(x,y) k_0 d_0]$ (2.4-11)

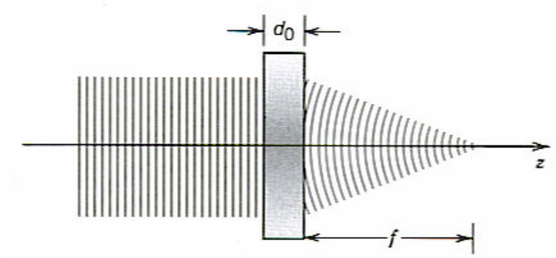

Figure 2.4-13 A graded-index plate acts as a lens.

2.5 Interference

Linearity of the wave equation → Superposition of the wavefunctions

But not superposition of the optical intensity because of interference.

(Consider waves of the same frequency in this section.)

A. Interference of Two Waves

$$U(\vec{r}) = U_1(\vec{r}) + U_2(\vec{r})$$

$$U_1 \equiv \sqrt{I_1} e^{j\varphi_1}, \quad U_2 \equiv \sqrt{I_2} e^{j\varphi_2}$$

Intensity interference equation:

$$I = |U|^2 = I_1 + I_2 + 2\sqrt{I_1 I_2} \cos\varphi$$

$$\varphi \equiv \varphi_2 - \varphi_1 \qquad (2.5\text{-}4,5)$$

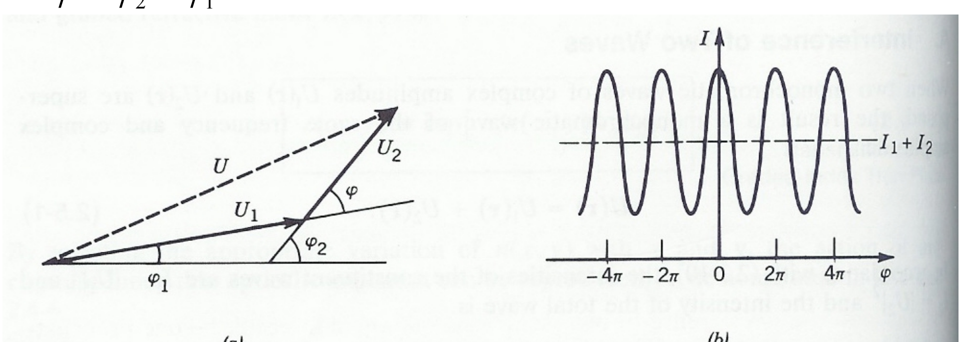

Figure 2.5-1 (*a*) Phasor diagram for the superposition of two waves of intensities I_1 and I_2 and phase difference $\varphi = \varphi_2 - \varphi_1$. (*b*) Dependence of the total intensity I on the phase difference φ.

Spatial redistribution of optical intensity. Power conservation still holds.

<u>Interferometer</u>

Waves superimpose with delay $d \rightarrow \varphi = 2\pi(d/\lambda)$

$$I = 2I_0\left[1+\cos\left(2\pi\frac{d}{\lambda}\right)\right] \qquad (2.5\text{-}6)$$

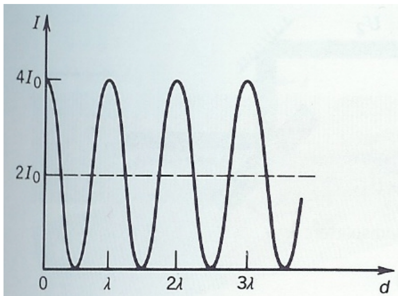

Figure 2.5-2 Dependence of the intensity I of the superposition of two waves, each of intensity I_0, on the delay distance d. When the delay distance is a multiple of λ, the interference is constructive; when it is an odd multiple of $\lambda/2$, the interference is destructive.

difference φ, the total intensity may be smaller than $I_1 + I_2$ at some positions and greater than $I_1 + I_2$ at others, with the total power (integral of the intensity) conserved.

Three important examples of intereferometers: Mach-Zehnder interferometer, Michelson interferometer, Sagnac interferometer.

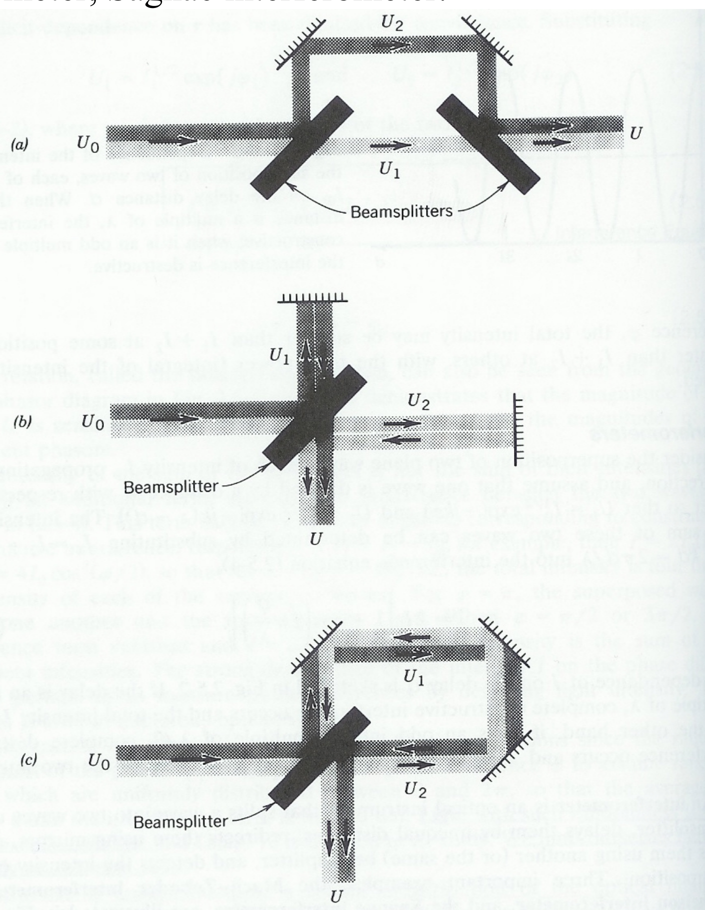

Figure 2.5-3 Interferometers: (*a*) Mach–Zehnder interferometer; (*b*) Michelson interferometer; (*c*) Sagnac interferometer. A wave U_0 is split into two waves U_1 and U_2. After traveling through different paths, the waves are recombined into a superposition wave $U = U_1 + U_2$ whose intensity is recorded. The waves are split and recombined using beamsplitters. In the Sagnac interferometer the two waves travel through the same path in opposite directions.

Intensity I is a very sensitive function of $\varphi = 2\pi \frac{d}{\lambda} = 2\pi \frac{nd}{\lambda_0} = 2\pi \frac{nfd}{c_0}$

→ Can measure small variation of d, n, λ_0, or f

<u>Interference of two oblique plane waves</u>

$$U_1 = \sqrt{I_0} \exp(-jkz)$$

$$U_2 = \sqrt{I_0} \exp[-jk(z\cos\theta + x\sin\theta)]$$

At $z = 0, \varphi = kx\sin\theta$,

$$\rightarrow \quad I = 2I_0[1 + \cos(kx\sin\theta)] \qquad (2.5\text{-}7)$$

Period of interference pattern $\Lambda = \lambda / \sin\theta$

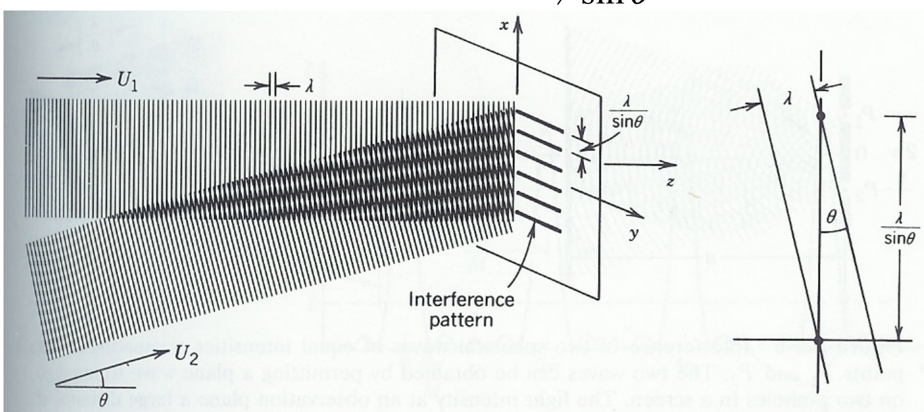

Figure 2.5-4 The interference of two plane waves at an angle θ results in a sinusoidal intensity pattern of period $\lambda/\sin\theta$.

B. Multiple-wave Interference

<u>M waves of equal amplitude and equal phase difference</u>

$$U_m = \sqrt{I_0} \exp[j(m-1)\varphi], \quad m = 1, 2, ..., M \qquad (2.5\text{-}9)$$

$$U = \sum_{m=1}^{M} U_m$$

$$I = |U|^2 = I_0 \frac{\sin^2(M\varphi/2)}{\sin^2(\varphi/2)} \qquad (2.5\text{-}10)$$

In the graph of I as a function of φ, the number of minor peaks between the main peaks = (M - 1).

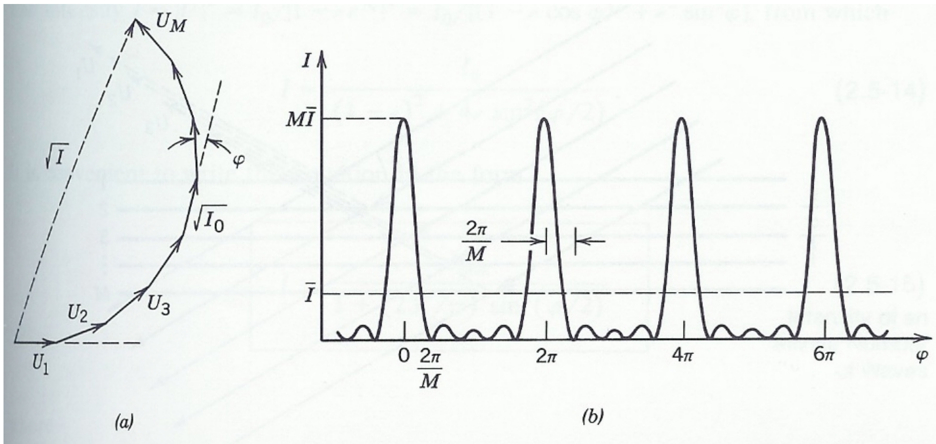

Figure 2.5-7 (*a*) The sum of M phasors of equal magnitudes and equal phase differences. (*b*) The intensity I as a function of φ. The peak intensity occurs when all the phasors are aligned; it is M times greater than the mean intensity $\bar{I} = MI_0$. In this graph $M = 5$.

<u>Infinite number of waves of progressively smaller amplitudes and equal phase difference</u>

$$U_m = \sqrt{I_0}\, r^{(m-1)} e^{j(m-1)\varphi}, \quad m = 1, 2, 3, \ldots$$

$$r < 1$$

$$U = \sum_{m=1}^{\infty} U_m = \frac{\sqrt{I_0}}{1 - re^{j\varphi}} \tag{2.5-13}$$

$$I = |U|^2 = \frac{I_{\max}}{1 + (2\mathscr{F}/\pi)^2 \sin^2(\varphi/2)} \tag{2.5-15}$$

where

$$I_{\max} \equiv \frac{I_0}{(1-r)^2} \tag{2.5-16}$$

$$\mathscr{F} \equiv \frac{\pi\sqrt{r}}{1-r} \quad \text{: Finesse} \tag{2.5-17}$$

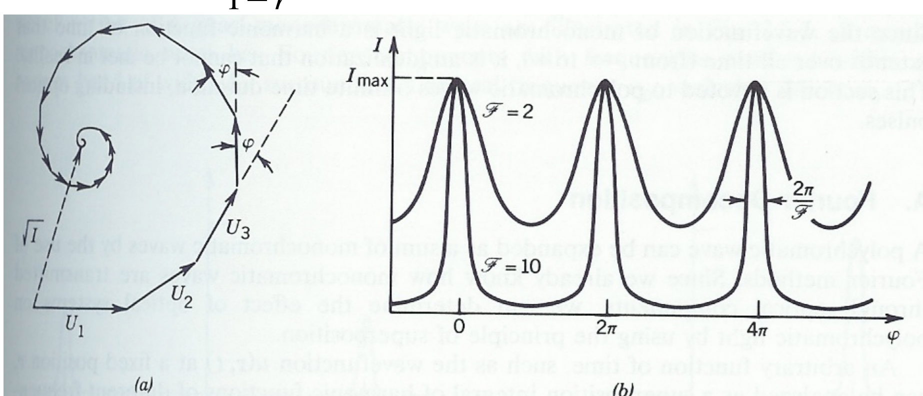

Figure 2.5-9 (*a*) The sum of an infinite number of phasors whose magnitudes are successively reduced at a geometric rate and whose phase differences φ are equal. (*b*) Dependence of the intensity I on the phase difference φ for two values of $\mathscr{F}$. Peak values occur at $\varphi = 2\pi q$. The width (FWHM) of each peak is approximately $2\pi/\mathscr{F}$ when $\mathscr{F} \gg 1$. The sharpness of the peaks increases with increasing $\mathscr{F}$.

When r approaches 1, $I_{\max}$ can be very large! → Principle of optical resonators, lasers.

- Physical meaning of $\mathcal{F}$

  Consider values of φ near the $\varphi = 0$ resonance peak

$$I \approx \frac{I_{\max}}{1+(\mathcal{F}/\pi)^2\varphi^2} \qquad (2.5\text{-}18)$$

Full width at half maximum (FWHM)

$$\Delta\varphi = \frac{2\pi}{\mathcal{F}} \qquad (2.5\text{-}19)$$

→ Finesse is a measure of the sharpness of the interference function.

2.6 Polychromatic Light

Can be expressed as the sum of monochromatic waves over frequency.

<u>The pulsed plane wave</u>

$$U(\vec{r},t) = \frac{1}{2\pi}\int_0^{\infty} A_\omega e^{-jkz} e^{j\omega t} d\omega$$
$$= a(t-\frac{z}{c}) \qquad (2.6\text{-}8,9)$$

$$a(t) = \frac{1}{2\pi}\int_0^{\infty} A_\omega e^{j\omega t} d\omega \qquad (2.6\text{-}10)$$

is a function with arbitrary shape

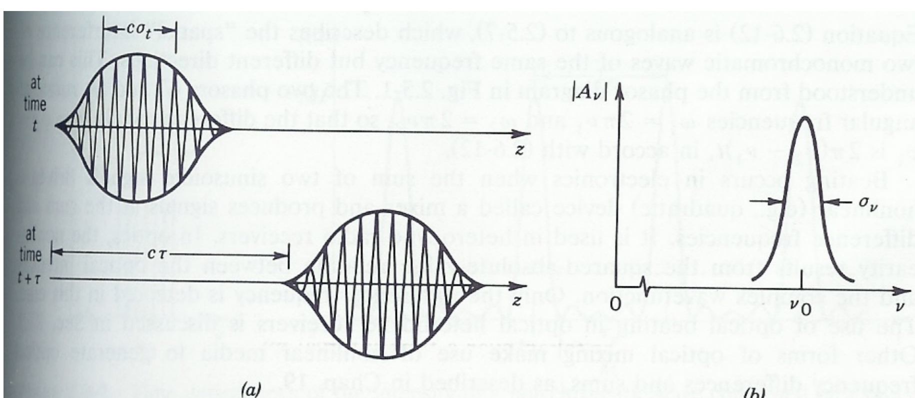

Figure 2.6-2 (*a*) The wavefunction $u(\mathbf{r},t) = \mathrm{Re}\{a(t - z/c)\}$ of a pulsed plane wave of time duration σ_t at times t and $t+\tau$. The pulse travels with speed c and occupies a distance $\sigma_z = c\sigma_t$. (*b*) The magnitude $|A_\nu|$ of the Fourier transform of the wavefunction is centered at ν_0 and has a width σ_ν.

If $a(t)$ is of finite duration σ_t in time →

Pulse width in space = $c\sigma_t$

Pulse width in frequency domain (spectral bandwidth) depends on pulse shape, see Appendix A.
E.g., for Gaussian function in space, its Fourier transform is still Gaussian in frequency domain. $\sigma_t = 1$ ps, $c\sigma_t = 0.3$ mm, $\sigma_f = \dfrac{1}{4\pi\sigma_t} = 80$ GHz.

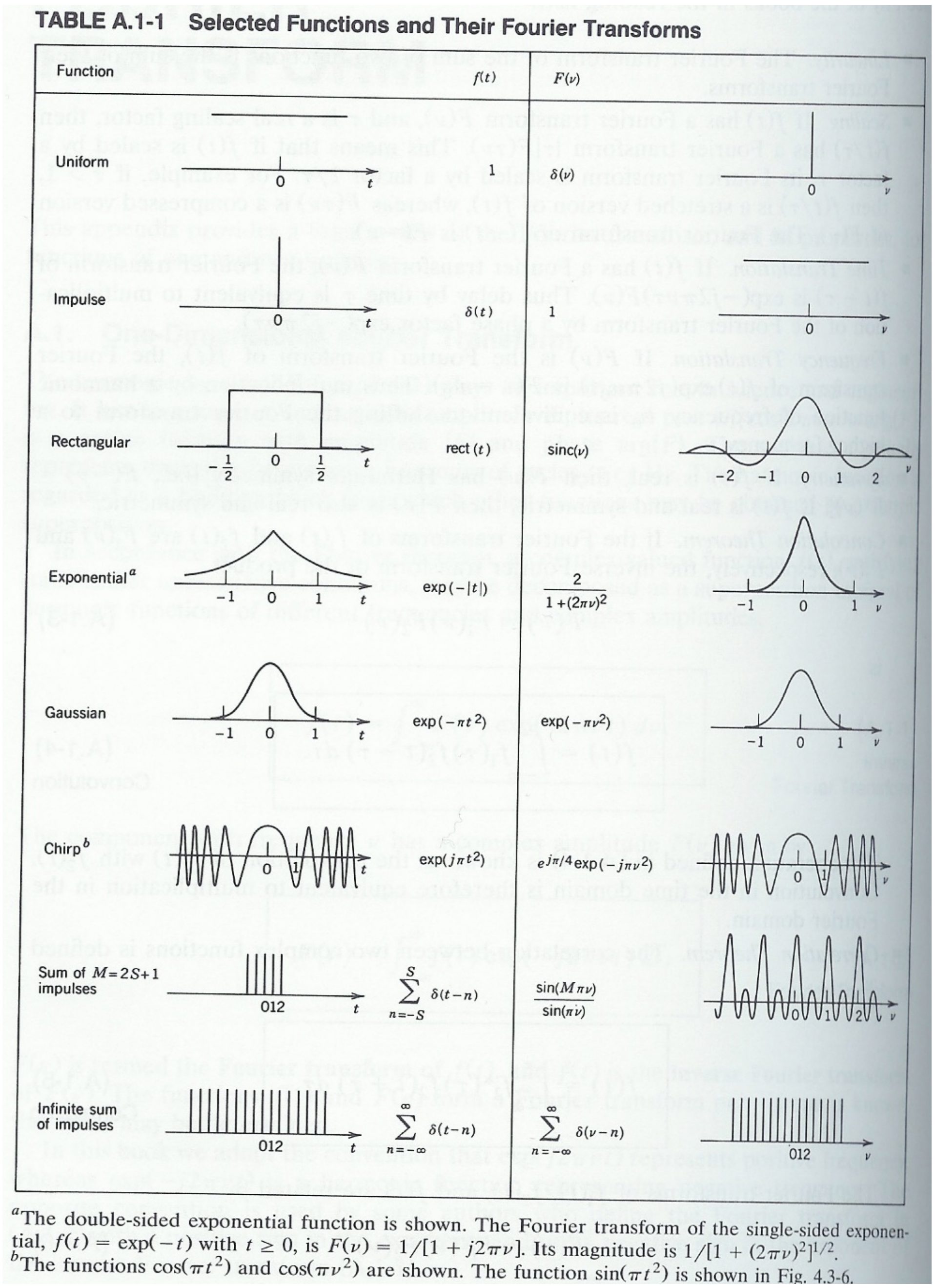

TABLE A.1-1 Selected Functions and Their Fourier Transforms

Function	$f(t)$	$F(\nu)$
Uniform	1	$\delta(\nu)$
Impulse	$\delta(t)$	1
Rectangular	$\mathrm{rect}(t)$	$\mathrm{sinc}(\nu)$
Exponential[a]	$\exp(-\lvert t\rvert)$	$\dfrac{2}{1+(2\pi\nu)^2}$
Gaussian	$\exp(-\pi t^2)$	$\exp(-\pi\nu^2)$
Chirp[b]	$\exp(j\pi t^2)$	$e^{j\pi/4}\exp(-j\pi\nu^2)$
Sum of $M=2S+1$ impulses	$\sum_{n=-S}^{S}\delta(t-n)$	$\dfrac{\sin(M\pi\nu)}{\sin(\pi\nu)}$
Infinite sum of impulses	$\sum_{n=-\infty}^{\infty}\delta(t-n)$	$\sum_{n=-\infty}^{\infty}\delta(\nu-n)$

[a] The double-sided exponential function is shown. The Fourier transform of the single-sided exponential, $f(t) = \exp(-t)$ with $t \geq 0$, is $F(\nu) = 1/[1 + j2\pi\nu]$. Its magnitude is $1/[1 + (2\pi\nu)^2]^{1/2}$.
[b] The functions $\cos(\pi t^2)$ and $\cos(\pi\nu^2)$ are shown. The function $\sin(\pi t^2)$ is shown in Fig. 4.3-6.

<u>Interference (beating) between two monochromatic waves</u>

$$U(t) = \sqrt{I_1}e^{j2\pi f_1 t} + \sqrt{I_2}e^{j2\pi f_2 t} \tag{2.6-11}$$

Utilizing the interference equation (2.5-4),

$$I = |U|^2 = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos[2\pi(f_1 - f_2)t] \qquad (2.6\text{-}12)$$

$$|f_1 - f_2| \equiv \text{Beat frequency}$$

<u>Interference of M monochromatic waves</u>

Consider an odd number M = 2L + 1 waves, each with intensity I_0 and frequencies

$$f_q = f_0 + qf_F, \quad q = \text{-L}, \ldots, 0, \ldots \text{L}$$

centered about f_0 and spaced by $f_F << f_0$.

$$U(t) = \sqrt{I_0}\sum_{q=-L}^{L} \exp[j2\pi(f_0 + qf_F)t] \qquad (2.6\text{-}13)$$

Utilizing Eq. (2.5-10),

$$I = |U|^2 = I_0 \frac{\sin^2(M\pi t/T_F)}{\sin^2(\pi t/T_F)} \qquad (2.6\text{-}14)$$

→ A periodic sequence of pulses with period $T_F \equiv 1/f_F$, peak intensity $M^2 I_0$.

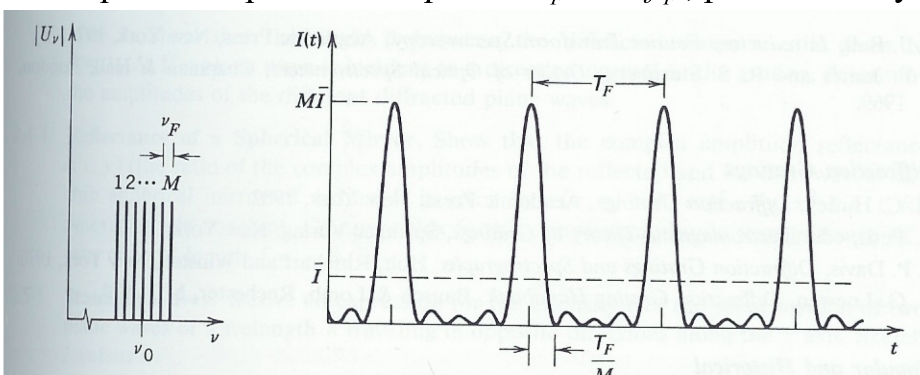

Figure 2.6-3 Time dependence of the intensity of a polychromatic wave composed of a sum of M monochromatic waves, of equal intensities, equal phases, and frequencies differing by ν_F. The intensity is a periodic train of pulses of period $T_F = 1/\nu_F$ with a peak M times greater than the mean. The duration of each pulse is M times smaller than the period. This should be compared with Fig. 2.5-7.

Application: Mode-locked laser for generating short laser pulses.